Part 1:

Let $N(R,B)$ denote the number of possible complete deals of the deck where the cumulative number of red cards **strictly** exceeds that of the cumulative number of black cards dealt at every point in time.

Let $D(R,B)$ denote the probability that the cumulative number of red cards strictly exceeds that of the cumulative number of black cards dealt at every point in time.

The total number of ways to deal R red cards and B black cards is $\binom{R+B}{R}$.

Then,

$\binom{R+B}{R} = N(R,B)$ + Number of possible deals of the deck where the cumulative number of red cards equals the cumulative number of black cards at some point in time, given that a red card is drawn as the first card + Number of possible deals of the deck where the black card is drawn as the first card.

The third component of the above is $\binom{R+B-1}{R}$.

Consider a dealing that satisfies the condition covered by the second component. Now switch the cards up to the point that the cumulative number of red cards equals the cumulative number of black cards. I.e. for the case $R = 5, B = 3$, one such dealing would be RRBBRBRR. Applying the switch yields BBRRRBRR.

There is therefore a one to one mapping between dealings satisfying the condition covered by the second component, and the dealings satisfying the condition covered by the third component.

So we have $N(R,B) = \binom{R+B}{R} - 2\binom{R+B-1}{R} = \binom{R+B}{R}\left(\frac{R-B}{R+B}\right)$.

This yields the desired probability $D(R,B) = \frac{R-B}{R+B}$.

We note that the above satisfies the recursion $D(R,B) = \frac{R}{R+B}D(R-1,B) + \frac{B}{R+B}D(R,B-1)$.

Part 2:

For this part, I have opted to approach it via iterating the recursion (the neat approach in part 1 doesn't seem to generalise easily).

Let $N(H,S,C)$ denote the number of possible complete deals of the deck where the cumulative number of heads dealt **strictly** exceeds **both** the cumulative number of spade cards dealt and the cumulative number of clubs card dealt.

Then,

$$N(H,S,C) = N(H-1,S,C) + N(H,S-1,C) + N(H,S,C-1).$$

Also note that

$$N(a,b,c) = N(a,c,b)$$
$$N(a,a,b) = N(a,b,a) = N(a,a,a) = 0\ (strict\ exceedance)$$

for non-negative integers a, b and c.

We want to find the probability that, given 16 cards, with 6 cards being hearts, 5 cards being spades and 5 cards being clubs, that, the cumulative number of heads dealt **strictly** exceeds **both** the

cumulative number of spade cards dealt and the cumulative number of clubs card dealt at each deal. That is, we want to find the probability

$$D(6,5,5) \coloneqq \frac{N(6,5,5)}{Total\ number\ of\ ways\ of\ arranging\ the\ given\ deck} = \frac{N(6,5,5)}{\frac{16!}{6!5!5!}}.$$

Now we iterate:

$$N(1,1,1) = 0$$
$$N(2,1,1) = N(1,1,1) + N(2,0,1) + N(2,1,0) = 2N(2,1,0)$$

$N(2,1,0)$ is exactly equivalent to the problem in part 1, with 2 red cards and 1 black card.

So $N(2,1,1) = 2N(2,1) = 2\left(\frac{3!}{2!1!}\right)\left(\frac{2-1}{2+1}\right) = 2.$

Continuing the iteration:

$$N(3,1,1) = N(2,1,1) + N(3,0,1) + N(3,1,0) = 2 + 2N(3,1) = 6$$
$$N(4,1,1) = N(3,1,1) + N(4,0,1) + N(4,1,0) = 6 + 2N(4,1) = 12$$
$$N(5,1,1) = N(4,1,1) + N(5,0,1) + N(5,1,0) = 12 + 2N(5,1) = 20$$
$$N(6,1,1) = N(5,1,1) + N(6,0,1) + N(6,1,0) = 20 + 2N(6,1) = 30$$
$$N(3,2,1) = N(2,2,1) + N(3,1,1) + N(3,2,0) = 0 + 6 + N(3,2) = 8$$
$$N(4,2,1) = N(3,2,1) + N(4,1,1) + N(4,2,0) = 8 + 12 + N(4,2) = 25$$
$$N(5,2,1) = N(4,2,1) + N(5,1,1) + N(5,2,0) = 25 + 20 + N(5,2) = 54$$
$$N(6,2,1) = N(5,2,1) + N(6,1,1) + N(6,2,0) = 54 + 30 + N(6,2) = 98$$
$$N(4,3,1) = N(3,3,1) + N(4,2,1) + N(4,3,0) = 0 + 25 + N(4,3) = 30$$
$$N(5,3,1) = N(4,3,1) + N(5,2,1) + N(5,3,0) = 30 + 54 + N(5,3) = 98$$
$$N(6,3,1) = N(5,3,1) + N(6,2,1) + N(6,3,0) = 98 + 98 + N(6,3) = 224$$
$$N(5,4,1) = N(4,4,1) + N(5,3,1) + N(5,4,0) = 0 + 98 + N(5,4) = 112$$
$$N(6,4,1) = N(5,4,1) + N(6,3,1) + N(6,4,0) = 112 + 224 + N(6,4) = 378$$
$$N(6,5,1) = N(5,5,1) + N(6,4,1) + N(6,5,0) = 0 + 378 + N(6,5) = 420$$
$$N(3,2,2) = N(2,2,2) + N(3,1,2) + N(3,2,1) = 0 + 2N(3,2,1) = 16$$
$$N(4,2,2) = N(3,2,2) + N(4,1,2) + N(4,2,1) = 16 + 2N(4,2,1) = 66$$
$$N(5,2,2) = N(4,2,2) + N(5,1,2) + N(5,2,1) = 66 + 2N(5,2,1) = 174$$
$$N(6,2,2) = N(5,2,2) + N(6,1,2) + N(6,2,1) = 174 + 2N(6,2,1) = 370$$
$$N(4,3,2) = N(3,3,2) + N(4,2,2) + N(4,3,1) = 0 + 66 + 30 = 96$$
$$N(5,3,2) = N(4,3,2) + N(5,2,2) + N(5,3,1) = 96 + 174 + 98 = 368$$
$$N(6,3,2) = N(5,3,2) + N(6,2,2) + N(6,3,1) = 368 + 370 + 224 = 962$$
$$N(5,4,2) = N(4,4,2) + N(5,3,2) + N(5,4,1) = 0 + 368 + 112 = 480$$
$$N(6,4,2) = N(5,4,2) + N(6,3,2) + N(6,4,1) = 480 + 962 + 378 = 1{,}820$$
$$N(6,5,2) = N(5,5,2) + N(6,4,2) + N(6,5,1) = 0 + 1{,}820 + 420 = 2{,}240$$
$$N(4,3,3) = N(3,3,3) + N(4,2,3) + N(4,3,2) = 0 + 2N(4,3,2) = 192$$
$$N(5,3,3) = N(4,3,3) + N(5,2,3) + N(5,3,2) = 192 + 2N(5,3,2) = 928$$
$$N(6,3,3) = N(5,3,3) + N(6,2,3) + N(6,3,2) = 928 + 2N(6,3,2) = 2{,}852$$
$$N(5,4,3) = N(4,4,3) + N(5,3,3) + N(5,4,2) = 0 + 928 + 480 = 1{,}408$$
$$N(6,4,3) = N(5,4,3) + N(6,3,3) + N(6,4,2) = 1{,}408 + 2{,}852 + 1{,}820 = 6{,}080$$
$$N(6,5,3) = N(5,5,3) + N(6,4,3) + N(6,5,2) = 0 + 6{,}080 + 2{,}240 = 8{,}320$$
$$N(5,4,4) = N(4,4,4) + N(5,3,4) + N(5,4,3) = 0 + 2N(5,4,3) = 2{,}816$$
$$N(6,4,4) = N(5,4,4) + N(6,3,4) + N(6,4,3) = 2{,}816 + 2N(6,4,3) = 14{,}976$$
$$N(6,5,4) = N(5,5,4) + N(6,4,4) + N(6,5,3) = 0 + 14{,}976 + 8{,}320 = 23{,}296$$
$$N(6,5,5) = N(5,5,5) + N(6,4,5) + N(6,5,4) = 0 + 2N(6,5,4) = 46{,}592.$$

The probability required is therefore $D(6,5,5) = \frac{46{,}592}{2{,}018{,}016} = \frac{16}{693}$.